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Mathematical and Physical Foundations of Quantum Statistics

Milan Vinduška[1]

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It is shown that the peculiar properties of quantum statistics can be explained as a consequence of the destroyed isotropy of the subquantum space.

1. INTRODUCTION

A certain progress has been achieved in the understanding of quantum phenomena such as EPR correlations. It appears that the experimentally studied Bell-like inequalities do not prove the nonlocality of quantum mechanics (or that of hidden variables), but rather the fact that the subquantum world cannot be treated as a metric space with constant curvature (Vinduška, 1993b, 1994b).

Our attitude to the problem coincides partially with that of Pitowsky (1982), Gudder (1993), and others (Vujičić *et al.*, 1994). The main aim of this paper is to give deeper reasons for the use of the ideas of relativity in the interpretation and further development of quantum theory.

2. BELL'S INEQUALITIES AS A CONSEQUENCE OF THE ISOTROPY OF THE HIDDEN VARIABLE SPACE

In the usual Bell scheme of hidden variables the correlation function has the form (Bell, 1964)

$$P(\mathbf{a}, \mathbf{b}) = \int A(\mathbf{a}, \lambda)B(\mathbf{b}, \lambda)\rho(\lambda)\, d\lambda \qquad (1)$$

where **a** and **b** represent space elements which characterize measured variables (see also Vinduška, 1994a, 1993a). $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ denote the experimental

[1] Compact Information Technology s.r.o., 160 00 Prague 6, Czech Republic.

outcomes for the first and the second particle, respectively (acquiring values of $+1$ and -1 according to convention), the λ's are hidden parameters, and $\rho(\lambda)$ is a positively defined, normalized measure of the probability.

In terms of the affine geometry the space elements **a** and **b** can be taken as unit vectors with their ends on a spherical surface and λ as radical rays with ends on some general surface **S**. In this case the probability density can be expressed as

$$\rho(\lambda)\, d\lambda \sim (\mathbf{n}\lambda)\, dS \tag{2}$$

Here **n** is a normal to the surface **S** such that $(\mathbf{n}\lambda) \geq 0$. The normalization condition then takes the form

$$\oint_S (\mathbf{n}\lambda)\, dS = 1 \tag{3}$$

The statistical scheme used is supposed to obey the following properties:

1. The isotropy of the macroscopic space in relation to the space elements defined by the measuring devices

$$P(\mathbf{a}, \mathbf{b}) = P(|\mathbf{a} - \mathbf{b}|) \tag{4}$$

2. The isotropy of the hidden variable space

$$A(a, \lambda), B(b, \lambda) \neq f(\mathbf{n}_1, \mathbf{n}_2, \ldots) \qquad \text{and} \qquad \rho(\lambda) \neq f(\mathbf{n}_1, \mathbf{n}_2, \ldots) \tag{5}$$

in relation to all vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots$ belonging to the space spanned by the vectors **a** and **b**.

It can be proved that the strongest correlations permitted by the considered Bell scheme of hidden variables are those for which the correlation function $P(\mathbf{a}, \mathbf{b})$ is a linear (sawlike) function of $|\mathbf{a} - \mathbf{b}|$ (Tyapkin and Vinduška, 1991).

The most straightforward proof of this statement can be given using the function

$$D(\mathbf{a}, \mathbf{b}) = \frac{1}{2P(\mathbf{a}, \mathbf{a})} \{P(\mathbf{a}, \mathbf{a}) - P(\mathbf{a}, \mathbf{b})\} \tag{6}$$

which plays the role of the metric distance in Bell's scheme of hidden variables, as follows from the following theorem.

Theorem. Let $D(0) = 0$, $D(\kappa) = 1$, and $A(\mathbf{a}, \lambda)$, $B(\mathbf{b}, \lambda)$, and $\rho(\lambda)$ guarantee the rotational invariance of $D(\mathbf{a}, \mathbf{b})$, i.e., $D(\mathbf{a}, \mathbf{b}) = D(\varphi_{ab})$. Then $D(\mathbf{a}, \mathbf{b})$ is the exact metric distance of the spherical or Riemannian geometry if and only if the sequence

$$A(a_1, \lambda), A(a_2, \lambda), \ldots, A(a_n, \lambda)$$

for an ordered set of the space elements

$$a_1, a_2, a_3, \ldots, a_n, \qquad \phi_{a_1 a_n} \leq \kappa$$

changes sign no more than once for each λ.

For a one-dimensional space of elements the ordering simply means placing $\mathbf{a}_i$ in correspondence with their increasing index; for a general spherical surface it means that $\mathbf{a}_i$ are placed similarly on the main circle. The κ denotes the distance for which $P(\kappa) = -P(0)$ holds. The proof of the theorem can be found, e.g., in Tyapkin and Vinduška (1991).

Example 1. The linear polarization of photon pairs can be described with hidden variables lying in the plane perpendicular to the photon momentum, using

$$A(\mathbf{a}, \lambda) = A(\mathbf{a}, \lambda_1) = \text{sign}(\lambda_1 \lambda_1^a) \tag{7}$$

$$B(\mathbf{a}, \lambda) = B(\mathbf{b}, \lambda_2) = \text{sign}(\lambda_2 \lambda_2^b) \tag{8}$$

The vector λ_i^a is constructed in such a way that $\mathbf{a}$ is a bisectrix of the angle $\angle \lambda_i$, $\mathbf{a}$. Hidden vectors are linked as $\lambda_1 \uparrow\uparrow \lambda_2$ or $\lambda_1 \perp \lambda_2$ for states with even and odd parity, respectively.

The sufficient condition for fulfilling the conditions of the theorem is the rotational invariance of $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$—which is guaranteed by (7) and (8)—and of $\rho(\lambda)$, which will hold if $\mathbf{S}$ in (2) and (3) is taken as a circle.

3. THE RELATIVE MEASURE OF PROBABILITY

The fact that the quantum correlation function is not a linear function of $|\mathbf{a} - \mathbf{b}|$ can be interpreted as experimental evidence that the hidden variable space is not isotropic.

Example 2. Let us suppose that the hidden variable space of linear photon polarization can be described with the use of Minkowski plane geometry, i.e., that our hidden variable scheme is invariant under hyperbolic rotations (Vinduška, 1991b). It is not difficult to rewrite (7) and (8) in terms of hyperbolic invariants

$$A(\mathbf{a}, \lambda) = A(\mathbf{a}, \lambda_1) = \text{sign}(\|\lambda_1^a\|_h^2) \tag{9}$$

$$B(\mathbf{b}, \lambda) = B(\mathbf{b}, \lambda_2) = \text{sign}(\|\lambda_2^b\|_h^2) \tag{10}$$

here $\|\lambda_i^r\|_h^2$ denotes the square of the hyperbolic norm of ith particle when axis $\mathbf{x}$ is identified with $\mathbf{r}$ ($\mathbf{r} = \mathbf{a}$ or $\mathbf{b}$).

A curve which defines the distribution of $\rho(\lambda)$ must be evidently constructed of four branches of hyperbolas and then

$$\rho(\lambda)\, d\lambda \to \|\lambda^{\mathbf{a}}\|_h^2 (4\|\lambda\|_e^2)^{-1} \tag{11}$$

where the subscript e denotes Euclidean norm.

Using (1) and (9)–(11), we get correct quantum mechanical results for the correlations.

The use of Minkowski geometry means a great step away from classical statistics because the coordinate system must be oriented in a definite way in relation to **a** or **b** in this case. We will reflect this fact with the additional index r

$$P_r(\mathbf{a}, \mathbf{b}) = \int A(a, \lambda) B(b, \lambda) \rho_r(\lambda)\, d\lambda \tag{12}$$

To fulfill the conditions of the macroscopic isotropy (4) we are forced to interpret as quantum mechanical correlations only such $P_r(\mathbf{a}, \mathbf{b})$ for which $\mathbf{r} = \mathbf{a}$ or $\mathbf{r} = \mathbf{b}$.

It appears, therefore, that the statistics with the relative measure (RM) is more restrictive than the classical one because we must abandon an absolute, independent measure ρ. This also the reason why the so-called no-go theorems for hidden variables are invalidated and why various proofs of nonlocality in quantum phenomena (Stapp, 1992; Hardy, 1992) can be considered as doubtful.

Starting from Tyapkin and Vinduška (1991), we have interpreted the RM as a necessity of only the relative description of the spacelike correlations. In such an interpretation the space elements defined by the measuring devices play a role of reference frames to which the description of the physical system must be related. Because of the equivalence of both particles and different space elements **a**, **b**, . . . there must exist a transformation which connects the different reference frames used

$$\rho_b(\lambda_2^b) = \hat{R}(\varphi_{ab}) \rho_a(\lambda_1^a) \tag{13}$$

The properties of the transformation of $\hat{R}(\varphi_{ab})$ can be deduced from the following reasonable assumptions (Vinduška, 1991a):

- A1. The principle of covariant description. In practice this means that all procedures of $A(a, \lambda^i)$, $B(b, \lambda^i)$, and also $\rho(\lambda^i)$ must have the same functional dependence on the corresponding variables λ^i.
- A2. One-to-one correspondence between $\lambda_1^a \Leftrightarrow \lambda_2^b$.
- A3. Each concrete event must be taken as an invariant of the transformation, i.e., the result of each concrete measurement must not depend on the reference frame used for its description.

Then the transformations of $\hat{R}(\varphi_{ab})$ can be found. It can be shown that they do not form a continuous group in relation to the parameter φ_{ab}, but a cyclic group only (the number of its elements is equal to the number of the mutually commuting operators). From this also certain peculiar properties of the quantum probabilities can be deduced (Pitowsky, 1991).

4. THE PROBLEM OF LOCALITY

During the golden age of the Bell inequalities in the last decades the possibility of exploiting the RM was rather disgarded because it seemed that it would bring nothing new in relation to the "nonlocal influences." This belief was based on the following seemingly convincing arguments:

(i) Any theory with RM must be nonlocal because the link between λ_1 and λ_2 leads to the change of the distribution $\rho(\lambda_2)$ when the λ_1 are measured.
(ii) Or the source must be able to guess the future orientations of the measuring devices for creating the correct ρ_a and ρ_b.
(iii) Or there is no "free will" and the orientations of **a** and **b** are determined by the source itself.

The actual situation can be clarified by introducing a simple classical but relativistic correlation (Vinduška, 1994b).

Example 3. The source **S** creates two particles 1 and 2 which bear hidden vectors λ_1 and λ_2 such that $\lambda_1 \uparrow\downarrow \lambda_2$ (see Fig. 1). For the whole ensemble of such particles the pairs of opposite vectors are equally distributed on the circle lying in the plane.

For the sake of simplicity the particles are supposed to be at rest in relation to the frame of reference related to the source when the measurements

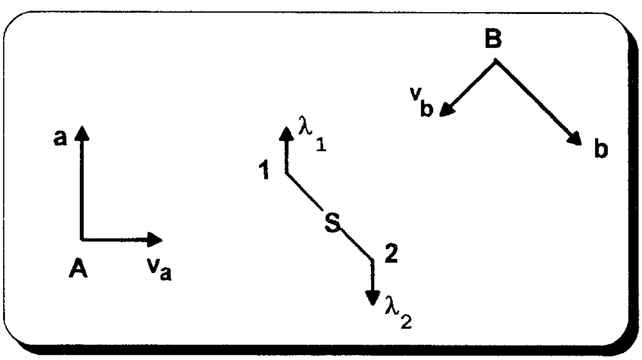

Fig. 1. The relativistic plane model for correlations.

are realized. The first experimenter using the device A is moving with velocity $\mathbf{v}_a$ in direction to the source and his apparatus is oriented perpendicularly to $\mathbf{a}$. He measures the value

$$A(\mathbf{a}, \lambda) = A(\mathbf{a}, \lambda_1) = \operatorname{sign}(\mathbf{a}\lambda_1)$$

where λ_1 is the hidden vector of the first particle.

Similarly, the second experimenter with device B moving with velocity $\mathbf{v}_b$ in another direction measures the value of the second particle onto $\mathbf{b}$ ($\mathbf{b} \perp \mathbf{v}_b$)

$$B(\mathbf{b}, \lambda) = B(\mathbf{b}, \lambda_2) = \operatorname{sign}(\mathbf{b}\lambda_2)$$

The measurements of both experimenters are realized in spacelike intervals and do not disturb each other. Let us remark that this plane model is reminiscent of a spin singlet system because $P(\mathbf{a}, \mathbf{a}) = -1$ and $P(\mathbf{a}, -\mathbf{a}) = +1$.

What distributions of λ_1 and λ_2 will be found by A and B when the measuring procedure is repeated many times? Taking into account the Lorentz transformations, it is not difficult to realize that the first experimenter will find

$$\rho_a(\lambda_1) = (1 - \beta^2)^{1/2}\left\{1 - \beta^2 \cos^2\left(\lambda_1 - \frac{\pi}{2}\right)\right\}^{-1}, \qquad 0 \leq \lambda_1 \leq \pi, \quad \beta = \mathbf{v}_a C^{-1}$$

i.e., the distribution of λ_1 has a peak around the direction of $\mathbf{a}$. Due to the property of the Lorentz transformation (the straight line on which λ_1 and λ_2 lie remains a straight line in any frame of reference), the distribution of λ_2 as is seen by the first experimenter is also peaked around the direction of $\mathbf{a}$ although there is no interaction between device A and the particle 2. Because the apparatuses are equivalent it is clear that the same reasoning relates to the second experimenter with device **B**. Here both distributions of λ_1 and λ_2 are peaked around $\mathbf{b}$.

Inspecting objections (i)–(iii) formulated above, we conclude that they fail in this purely classical example.

What we can say about the correlation function and Bell's inequalities in this case? Because each concrete event is invariant here, it is clear that $D(\mathbf{a}, \mathbf{b})$ is a metric distance of the spherical geometry on the circle and, therefore, the Bell inequalities are satisfied only if the angle (or the "distance") between $\mathbf{a}$ and $\mathbf{b}$ is measured in the reference frame where the source is at rest. This is not so in the case when this angle is measured with regard to the relation of one apparatus to the other because in such a case the hidden anisotropy of the whole picture appears.

Let us note that we have by this example an explanation of quantum phenomena not with the use of some hidden relativistic velocities, but rather by stressing the deep affinity between both great theories of 20th century physics.

5. CONCLUDING REMARKS

With regard to the role of quantum theory in human knowledge, physicists are divided into two groups.

The majority considers quantum mechanics as a more general theory than classical physics and insists that all physical phenomena can be (and must be) described in the language of QM. In this approach difficulties arise with different macroscopic phenomena such as Schrödinger's cat, the moon when nobody is looking at it, etc.

The minority believes, on contrary, that all quantum phenomena can be explained by classical statistics. This approach meets difficulties with "no-go" theorems for hidden variables.

What can we say about this problem from the point of view of the relative measure of the probability? At first sight it could seem that relative measure supports the first opinion because the probability measure ρ_a is more general than the absolute, independent ρ.

Nevertheless, the more thorough inspection of Examples 1 and 2 shows that this is not so. Here the difference between quantum statistics and classical statistics is as deep as the difference between Euclidean and Minkowski plane geometries. A similar conclusion can be drawn from the consideration of other correlations where the destroyed isotropy has another character.

It seems, therefore, that quantum statistics and classical statistics can be applied only in their own specific domains.

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